The University of Georgia

Department of Physics and Astronomy

Prelim Exam

August 11, 2025

Part I (Problems 1, 2, 3, and 4)

9:00 am – 1:00 pm

Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

Problem 1: Classical Mechanics (CM1)

A cannon shoots a ball at an angle θ_0 above the horizontal ground with an initial speed of v_0. Let $r(t)$ denote the ball's distance from the cannon. What is the largest possible value of θ_0 if $r(t)$ is to increase throughout the ball's flight? Assume that air resistance is negligible.

Note: $r(t)$ is the geometric distance from the launch point (the magnitude of the displacement vector), NOT the horizontal range.

Solution

Assuming that air resistance is negligible, the position of the cannon ball as a function of time is given by the standard x and y introductory physics equations for projectile motion:

$$x(t) = v_0 \cos(\theta_0)\, t$$
$$y(t) = v_0 \sin(\theta_0)\, t - \frac{1}{2} g t^2$$

Where v_0 is the initial velocity, $t = 0$ is the instant of launch, and the ball is launched from the origin.

The distance from the cannon as a function of time is then $r(t) = \sqrt{\left(x(t)\right)^2 + \left(y(t)\right)^2}$.

For the distance to be increasing as a function of time, the first derivative must be greater than zero. Because the square root function is monotonically increasing, maximizing the square root is equivalent to maximizing what is under the square root symbol, $\left(x(t)\right)^2 + \left(y(t)\right)^2$. Therefore, we can solve $\frac{d(r^2)}{dt} = 0$ instead, to simplify our math.

In that case, we have

$$\frac{d(r^2)}{dt} = 2(v_0 \cos\theta_0)(v_0 \cos\theta_0)t + 2\left((v_0 \sin\theta_0)t - \frac{1}{2} g t^2\right)(v_0 \sin\theta_0 - gt)$$

which simplifies to

$$\frac{d(r^2)}{dt} = g^2 t^3 - 3(v_0 g \sin\theta_0) t^2 + 2 v_0^2 t.$$

Factoring out t and setting the result to be at least zero, we have

$$\frac{d(r^2)}{dt} = t(g^2 t^2 - 3(v_0 g \sin\theta_0)t + 2v_0^2) \geq 0$$

This will be true provided that $g^2 t^2 - 3(v_0 g \sin\theta_0)t + 2v_0^2$ is at least zero. We use the quadratic formula to solve for the zeros and after simplifying, we find

$$t = \frac{v_0}{2g}\left(3\sin\theta_0 \pm \sqrt{9\sin^2\theta_0 - 8}\right)$$

For small θ_0, $r(t)$ will always increase because the argument of the square root is negative (and hence, $g^2t^2 - 3(v_0 g\sin\theta_0)t + 2v_0^2$ is always positive). The argument of the square root will become positive (and hence $g^2t^2 - 3(v_0 g\sin\theta_0)t + 2v_0^2$ will be negative) provided that $9\sin^2\theta_0 - 8 \geq 0$, and the cutoff occurs when $9\sin^2\theta_0 - 8 = 0$. Solving for θ_0, we find that

$$\sin\theta_0 = \sqrt{\frac{8}{9}} \;\Rightarrow\; \theta_0 = \sin^{-1}\left(\sqrt{\frac{8}{9}}\right)$$

when this happens. Therefore, provided $\theta_0 < \sin\left(\sqrt{8/9}\right)$, $r(t)$ will always be increasing.

Problem 2: Classical Mechanics (CM2)

Let's consider an object with mass m resting at a local equilibrium point (x_0) under the influence of potential $V(x)$. Due to a small perturbation, the object started oscillating around the equilibrium point.

a) Start with the Taylor expansion of $V(x)$ around the equilibrium point, show that the oscillation frequency is $\omega = \sqrt{V''(x_0)/m}$

b) If $V(x) = A/x^2 - B/x$, where $A, B > 0$, find the small oscillation frequency in terms of m, A, and B.

(Hint) For a simple harmonic motion under a potential, $V(x) = \frac{1}{2}k(x - x_0)^2$, the frequency of small oscillations is $\omega = \sqrt{k/m}$ where m is the object mass and k is spring constant.

Solution

a) $V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2!}V''(x_0)(x - x_0)^2 + \frac{1}{3!}V'''(x_0)(x - x_0)^3 + \cdots$

$V(x_0)$ is an additive constant that can be ignored and, for an equilibrium point, $V'(x_0) = 0$. Ignoring higher order terms in small oscillations,

$V(x) \approx \frac{1}{2}V''(x_0)(x - x_0)^2$

Using the provided hint, one can get the oscillation frequency of

$\omega = \sqrt{V''(x_0)/m}$

b) We need to find the equilibrium point first.

$V'(x_0) = -2A/x^3 + B/x^2 = 0 \rightarrow x = x_0 = 2A/B$

$V''(x) = 6A/x^4 - 2B/x^3$

plug in $x_0 = 2A/B$,

$$V''(x_0) = \frac{6A}{x_0^4} - \frac{2B}{x_0^3} = \frac{6A}{\left(\frac{2A}{B}\right)^4} - \frac{2B}{\left(\frac{2A}{B}\right)^3} = \frac{B^4}{8A^3}$$

$$\omega = \sqrt{V''(x_0)/m} = \sqrt{\frac{B^4}{8mA^3}}$$

Problem 3: Electricity and Magnetism (EM1)

A good conductor, with a conductivity σ and magnetic permeability μ, occupies the half-space $x > 0$; the region $x < 0$ is vacuum, with a time-dependent magnetic field $\vec{\mathbf{B}}(t) = B_0 \cos \omega t\ \hat{\mathbf{e}}_y$. You may ignore the displacement current for this problem.

a) Draw a clearly labeled figure that illustrates the problem. Write down the equations that describe the problem. To facilitate the solution, introduce complex exponentials to represent the trigonometric functions.
b) Solve the equations to obtain the behavior of the magnetic field for $x > 0$. Make sure to take the real part (i.e., express your result in terms of real functions). In your solution, identify the skin depth $\delta(\omega)$, and comment on its physical significance.

Note: you may find Ohm's Law, $\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$, useful for this problem.

Solution

a)

A quick sketch of the problem is straightforward. The equations are Ampère's Law (ignoring the displacement current) $\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}$; Ohm's Law, $\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$, with σ the conductivity; and $\vec{\mathbf{H}} = \vec{\mathbf{B}}/\mu$, with μ the permeability for the conductor. We then have

$$\nabla \times \vec{\mathbf{B}} = \sigma\mu\vec{\mathbf{E}} \tag{1}$$

Taking the curl of both sides, and using $\nabla \times \left(\nabla \times \vec{\mathbf{B}}\right) = \nabla\left(\nabla \cdot \vec{\mathbf{B}}\right) - \nabla^2\vec{\mathbf{B}} = -\nabla^2\vec{\mathbf{B}}$, along with Faraday's Law, $\nabla \times \vec{\mathbf{E}} = -\partial\vec{\mathbf{B}}/\partial t$, we obtain the vector diffusion equation

$$\nabla^2\vec{\mathbf{B}} = \frac{\mu\sigma\partial\vec{\mathbf{B}}}{\partial t}. \tag{2}$$

Specializing this to the geometry above, with $\vec{\mathbf{B}}(\vec{x}, t) = B(x)e^{-i\omega t}\hat{\mathbf{y}}$, we obtain

$$\frac{d^2B}{dx^2} = -i\omega\mu\sigma B. \tag{3}$$

b)

This is a linear second-order equation, with solutions of the form $e^{\lambda x}$; substituting into the differential equation, we find $\lambda^2 = -i\mu\sigma\omega$. Taking the square root, and recalling that $\sqrt{-i} = (1 - i)/\sqrt{2}$, we obtain

$$\lambda_\pm = \pm\frac{1 - i}{\delta(\omega)}, \tag{4}$$

where $\delta(\omega)$ is the skin depth,

$$\delta(\omega) = \sqrt{\frac{2}{\mu\sigma\omega}}. \tag{5}$$

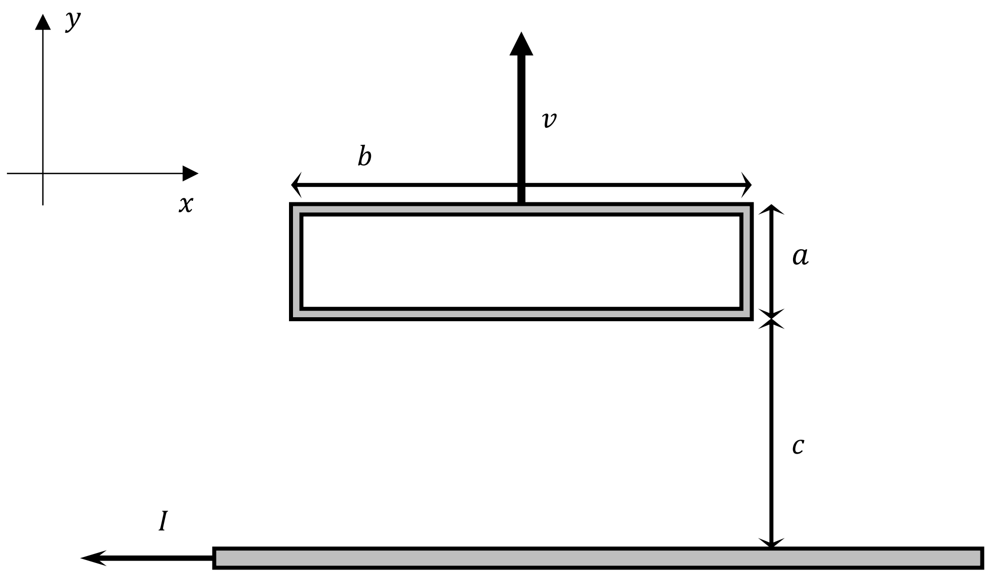
The skin depth provides a measure of the distance the ac magnetic field can penetrate into the conductor. Of the two solutions, the one corresponding to λ_+ diverges exponentially for large x, so we set the coefficient of that term equal to zero; after applying the boundary condition at $x = 0$, we obtain

$$\begin{aligned} B(x,t) &= \mathrm{Re}\left[B_0 e^{-(1-i)x/\delta} e^{-i\omega t}\right] \\ &= B_0 e^{-x/\delta} \cos(x/\delta - \omega t)\,. \end{aligned} \tag{6}$$

Problem 4: Electricity and Magnetism (EM2)

The rectangular loop and the wire shown in the figure both lie in the plane of the page. The loop has dimensions of $a = 1.00$ cm by $b = 4.00$ cm and has a resistance of $R = 0.0200\ \Omega$. The loop is moving directly away from the wire (in the plane of the page) at $v = 10.0$ m/s. The wire carries a current of 15.0 A to the left. What is the induced current (magnitude and direction) in the loop at the instant that $c = 2.00$ cm?

Hint: Find the flux through the loop as a function of distance from the wire.



Solution

First, let's find the flux through the loop as a function of its distance from the wire. Let $a = 1.0\ cm$ and $b = 4.0\ cm$ be the dimensions of the loop, and let $c = 2.0\ cm$ be the distance between the wire and the near edge of the loop. The magnetic field created by the wire is cylindrically symmetric, and does not vary along the x-axis. Let's set up an integral along the y-axis to find the flux. The field is parallel to the normal vector of the loop where is crosses through the loop. An infinitesimal element of flux through the loop is

$$d\Phi_B = \vec{B} \cdot d\vec{A} = B\ dA = B(b\ dy) = \frac{\mu_0 I b}{2\pi y} dy$$

Now we integrate this across the loop

$$\Phi_B = \frac{\mu_0 I b}{2\pi} \int_c^{c+a} \frac{dy}{y} = \frac{\mu_0 I b}{2\pi} \ln y \Big|_c^{c+a} = \frac{\mu_0 I b}{2\pi} \ln\left(\frac{c+a}{c}\right)$$

If desired, we can rewrite this in terms of y:

$$\Phi_B = \frac{\mu_0 I b}{2\pi} \ln\left(\frac{y+a}{y}\right)$$

Now, to find the induced current, we need to find the induced emf as the loop moves away from the wire. For this, we can use Faraday's law

$$|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right|$$

Note that

$$\frac{d\Phi_B}{dt} = \frac{d\Phi_B}{dy}\frac{dy}{dt} = \frac{d\Phi_B}{dy} v, \text{where } v = \frac{dy}{dt} \text{ is the speed of the loop.}$$

Thus the emf is

$$|\mathcal{E}| = \left|\frac{d\Phi_B}{dy}\right| v = \left|\frac{\mu_0 I b v}{2\pi}\frac{d}{dy}\left[\ln\left(\frac{y+a}{y}\right)\right]\right| = \left|\frac{\mu_0 I b v}{2\pi}\left(\frac{y}{y+a}\right)\left(\frac{y-(y+a)}{y^2}\right)\right|$$
$$= \frac{\mu_0 I b v}{2\pi}\left(\frac{a}{y(y+a)}\right)$$

Plugging in values, the induced emf at the instant shown is

$$|\mathcal{E}| = \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(15\ \mathrm{A})(4.0\ \mathrm{cm})(10\ \mathrm{m/s})}{2\pi} \frac{1.0\ \mathrm{cm}}{(2.0\ \mathrm{cm})(3.0\ \mathrm{cm})} = 2.0 \times 10^{-5}\ \mathrm{V}$$

Finally, the induced current through the loop is thus

$$I = \frac{\mathcal{E}}{R} = \frac{2.0 \times 10^{-5}\ \mathrm{V}}{0.020\ \Omega} = 0.0010\ \mathrm{A} = 1.0\ \mathrm{mA}$$

The field points into the page through the loop, and is decreasing in magnitude as the loop moves away from the wire, so Lenz's law tells us that the induced current will flow clockwise.