The University of Georgia

Department of Physics and Astronomy

Prelim Exam

August 11, 2025

Part II (Problems 5 and 6)

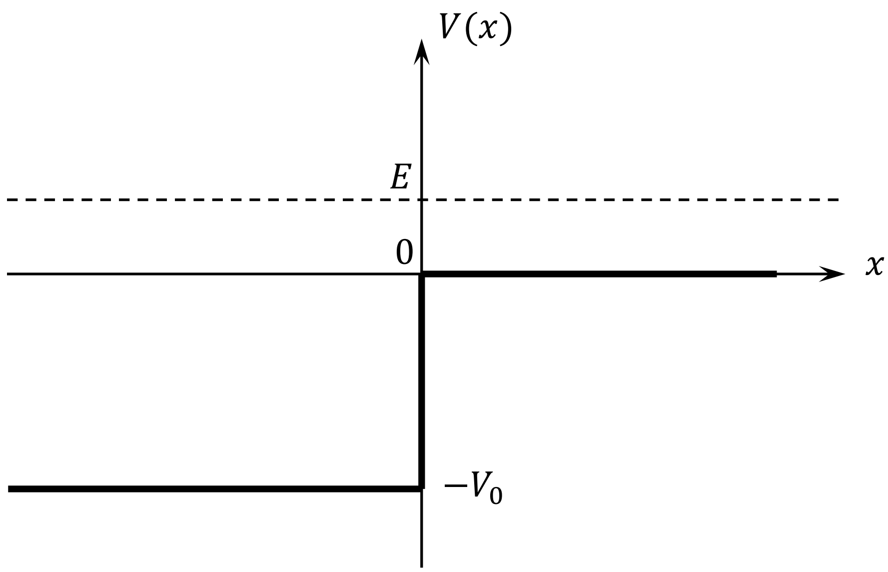
3:00 pm – 5:00 pm

Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

Problem 5: Quantum Mechanics (QM1)

When studying electron emission by metals, it is necessary to take into account the fact that electrons with energy sufficient to leave the metal, according to quantum mechanics, can be reflected from the metal boundary. Considering a one-dimensional model with potential $V(x) = -V_0$ at $x < 0$ (inside the metal) and $V(x) = 0$ at $x > 0$ (outside the metal), determine the reflection coefficient of an electron with energy $E > 0$ from the metal surface ($x = 0$).



Solution

For $x < 0$ the Schrödinger equation and the corresponding solution are

$$\frac{d^2\psi_1}{dx^2} + k^2\psi_1 = 0, \;\; \psi_1 = Ae^{ikx} + Be^{-ikx}, \;\; k^2 = \frac{2m}{\hbar^2}(E + V_0)$$

while for $x > 0$

$$\frac{d^2\psi_2}{dx^2} + K^2\psi_2 = 0, \;\; \psi_2 = Ce^{iKx}, \;\; K^2 = \frac{2m}{\hbar^2}E$$

The reflection coefficient, R, by definition is

$$R \equiv \left|\frac{B}{A}\right|^2$$

Continuity of ψ and ψ' at $x = 0$ results in the following two equations,

$$A + B = C$$

and

$$ik(A - B) = iKC$$

from where

$$\frac{B}{A} = \frac{k - K}{k + K}$$

and thus,

$$R \equiv \left|\frac{k - K}{k + K}\right|^2 = \frac{\left(\sqrt{E + V_0} - \sqrt{E}\right)^2}{\left(\sqrt{E + V_0} + \sqrt{E}\right)^2} = \frac{{V_0}^2}{\left(\sqrt{E + V_0} + \sqrt{E}\right)^4}$$

<u>Remark</u>: Notice that at large energies, $E \gg V_0$,

$$R \approx \frac{{V_0}^2}{16E^2}$$

dropping rapidly as the energy increases, which is expected on physical grounds. In the other limiting case, $E \ll V_0$,

$$R \approx 1 - 4\sqrt{\frac{E}{V_0}}$$

For normal metals, $V_0 \approx 10$ eV, which for $E = 0.1$ eV gives $R = 0.67$.

Problem 6: Quantum Mechanics (QM2)

The Hamiltonian for a charged particle of charge q and mass m in the presence of a vector potential $\boldsymbol{A}$ is

$$H = \frac{\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2}{2m}.$$

Show that the probability flux $\boldsymbol{j}$, given by

$$\boldsymbol{j} = \mathrm{Re}\left(\psi^* \frac{\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)}{m}\psi\right),$$

satisfies the continuity equation

$$\frac{\partial}{\partial t}|\psi|^2 + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0.$$

Solution

Flux is

$$\boldsymbol{j} = \frac{\hbar}{2im}\left(\psi^*\,\boldsymbol{\nabla}\,\psi - \psi\,\boldsymbol{\nabla}\,\psi^* - \frac{2iq}{\hbar c}\boldsymbol{A}\,|\psi|^2\right),$$

$$\frac{\partial\psi}{\partial t} = \frac{-i}{2m\hbar}\left[-\hbar^2\nabla^2\psi + \frac{2i\hbar q}{c}A\cdot(\nabla\psi) + \frac{2i\hbar q}{c}\,(\nabla\cdot A)\psi + \frac{q^2A^2}{c^2}\psi\right].$$

$$\frac{\partial}{\partial t}|\psi|^2 + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0.$$