The University of Georgia
Department of Physics and Astronomy

Prelim Exam
August 9, 2024

Part I (Problems 1, 2, 3, and 4)
9:00 am – 1:00 pm

Instructions:

- Start each problem on a new sheet of paper. Write the problem number on the top left of each page and your pre-arranged prelim ID number (but ***not*** your name) on the top right of each page.
- Leave margins for stapling and photocopying.
- Write only on *one side* of the paper. Please ***do not*** write on the back side.
- If not advised otherwise, derive the mathematical solution for a problem from basic principles or general laws (Newton's laws, the Maxwell equations, the Schrödinger equation, etc.).
- You may use a calculator for basic operations only (i.e., not for referring to notes stored in memory, symbolic algebra, symbolic and numerical integration, etc.) The use of cell phones, tablets, and laptops is not permitted.
- Show your work and/or explain your reasoning in *all* problems, as the graders are not able to read minds. Even if your final answer is correct, not showing your work and reasoning will result in a *substantial* penalty.
- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

Problem 1: Classical Mechanics (CM1)

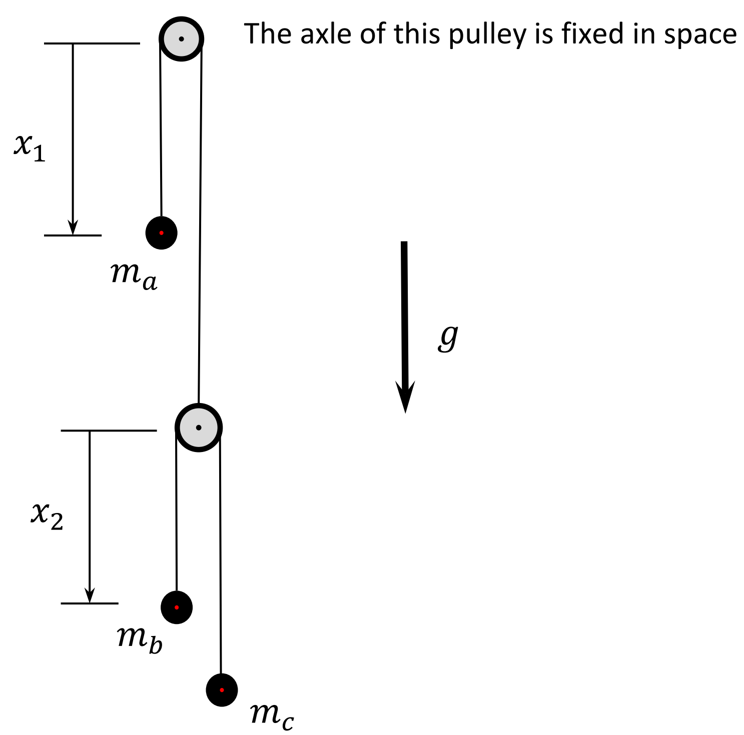
Consider the system of three masses m_a, m_b, m_c, shown as black circles in the figure below. The grey circles represent massless, frictionless, freely rotating pulleys of negligibly small radii. The axle of the upper pulley is held at fixed height. The axle of the lower pulley is connected to mass m_a by the upper string. Mass m_b is connected to mass m_c by the lower string. Both strings are massless and have fixed lengths, ℓ_1 and ℓ_2, respectively.

The three masses and the lower pulley are subject to the tension forces of their respective strings. The three masses are also subject to their respective weight forces, with gravitational acceleration g.

Choose variables x_1 and x_2 as depicted in the figure to be generalized coordinates.

a) Write down the Lagrangian for this system in terms of variables x_1 and x_2, their first time derivatives $\dot{x}_1$ and $\dot{x}_2$, and constants g, m_a, m_b, m_c, ℓ_1, and ℓ_2, as needed. [Note: do not invent alternative variables or constants to replace x_1 or x_2.]

b) Write down the Euler-Lagrange equations of motion for x_1 and x_2. [You do **not** need to solve these equations.]



SOLUTION to CM1

a) Let x_a, x_b, x_c and x_p be the vertical positions of the masses and the mobile pulley, all relative to the center of the fixed pulley. We need to write each of these in term of the generalized coordinates x_1 and x_2. We directly have $x_a = x_1$. Assuming the radii of the pulleys are negligible compared to ℓ_1 and ℓ_2, we can get the following relations:

$$\ell_1 = x_a + x_p, \qquad \ell_2 = (x_b - x_p) + (x_c - x_p)$$

The first equation lets us write

$$x_p = \ell_1 - x_a = \ell_1 - x_1$$

From the figure, we can see some relations:

$$x_b = x_p + x_2 = \ell_1 - x_1 + x_2$$

$$x_c = x_p + (\ell_2 - x_2) = \ell_1 - x_1 + \ell_2 - x_2$$

Now that we have each of the positions expressed in terms of the generalized coordinates, we can write the kinetic and potential energy expressions. First, kinetic energy:

$$\begin{aligned} T &= \frac{1}{2}m_a\dot{x}_a^2 + \frac{1}{2}m_b\dot{x}_b^2 + \frac{1}{2}m_c\dot{x}_c^2 \\ &= \frac{1}{2}m_a\dot{x}_1^2 + \frac{1}{2}m_b(-\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2}m_c(-\dot{x}_1 - \dot{x}_2)^2 \end{aligned}$$

Note that ℓ_1 and ℓ_2 are constant, so they drop out of the time derivatives. Now for the potential energy, taking the center of the fixed pulley as the zero and down as the positive direction:

$$\begin{aligned} V &= -m_a g x_a - m_b g x_b - m_c g x_c \\ &= -m_a g x_1 - m_b g(-x_1 + x_2 + \ell_1) - m_c g(-x_1 - x_2 + \ell_1 + \ell_2) \\ &= -(m_a - m_b - m_c)g x_1 - (m_b - m_c)g x_2 - (m_b + m_c)g\ell_1 - m_c g\ell_2 \end{aligned}$$

Combining these gives us the Lagrangian:

$$\begin{aligned} L = T - V = {} & \frac{1}{2}m_a\dot{x}_1^2 + \frac{1}{2}m_b(-\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2}m_c(-\dot{x}_1 - \dot{x}_2)^2 \\ & + (m_a - m_b - m_c)g x_1 + (m_b - m_c)g x_2 + (m_b + m_c)g\ell_1 + m_c g\ell_2 \end{aligned}$$

b) The Euler-Lagrange equation of motion for the ith generalized coordinate is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

For x_1, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\mathrm{d}}{\mathrm{d}t}[m_a\dot{x}_1 + m_b(\dot{x}_1 - \dot{x}_2) + m_c(\dot{x}_1 + \dot{x}_2)]$$
$$= m_a\ddot{x}_1 + m_b(\ddot{x}_1 - \ddot{x}_2) + m_c(\ddot{x}_1 + \ddot{x}_2)$$

$$\frac{\partial L}{\partial x_1} = (m_a - m_b - m_c)g$$

Thus, the equation of motion is

$$m_a\ddot{x}_1 + m_b(\ddot{x}_1 - \ddot{x}_2) + m_c(\ddot{x}_1 + \ddot{x}_2) = (m_a - m_b - m_c)g$$

For x_2, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\mathrm{d}}{\mathrm{d}t}[m_b(-\dot{x}_1 + \dot{x}_2) + m_c(\dot{x}_1 + \dot{x}_2)]$$
$$= m_b(-\ddot{x}_1 + \ddot{x}_2) + m_c(\ddot{x}_1 + \ddot{x}_2)$$

$$\frac{\partial L}{\partial x_2} = (m_b - m_c)g$$

Thus, the equation of motion is

$$m_b(-\ddot{x}_1 + \ddot{x}_2) + m_c(\ddot{x}_1 + \ddot{x}_2) = (m_b - m_c)g$$

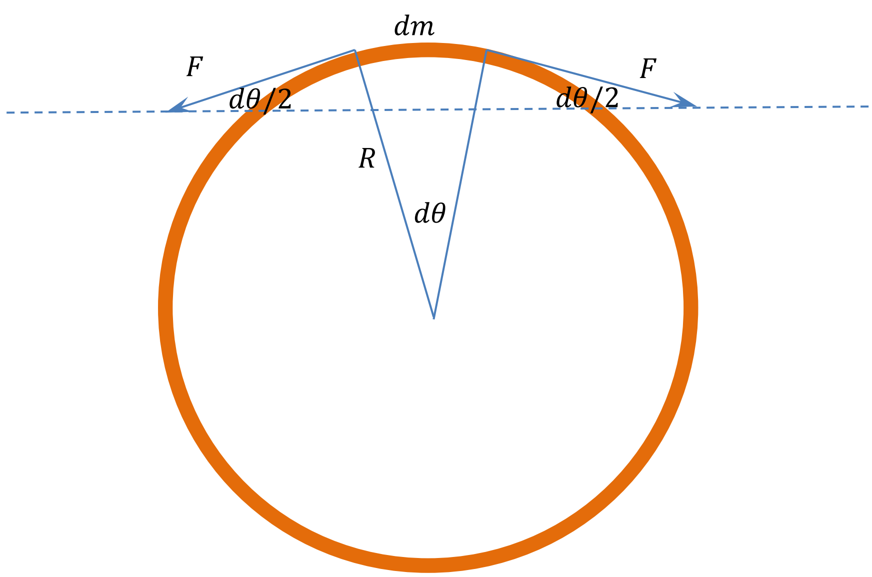
Problem 2: Classical Mechanics (CM2)

A circular ring of radius R_0 (when at rest) is made from a thin rubber band of mass m and stiffness k (elastic force per elongation length). The ring is spun around the axis passing through its center, perpendicular to the plane of the ring. Find the new radius R of the ring if the angular velocity of its rotation is ω. Express R in terms of R_0, k, m and ω. Ignore gravity.

SOLUTION I (Newtonian Mechanics):

In the inertial lab frame where ring is spinning at angular velocity ω:

Consider an infinitesimally small portion of mass dm at the top of the spinning ring:



Centripetal force acting on dm is $F_c = \frac{dmv^2}{R} = Fd\theta = k\ 2\pi(R - R_0)d\theta$, with $dm = \frac{md\theta}{2\pi} \ll m$, where we used the small angle approximation, $\sin x \approx x$, with x measured in radians. Considering that $v = \omega R$, after some algebra we find:

$$R = \frac{R_0}{1 - \frac{m\omega^2}{4\pi^2 k}}$$

SOLUTION II (Non-inertial frame of reference):

In the non-inertial, rotating reference frame where the ring is at rest:

Elastic potential energy of the ring, stretched from equilibrium radius $R_0$to radius R:

$$U_{\mathrm{E}}(R) = +\frac{1}{2}k[2\pi(R - R_0)]^2$$

Centrifugal force $F_{\mathrm{C}}(R) = m\omega^2 R$ contributes a centrifugal potential energy

$$U_C(R) = -\int_0^R F_C(R')\, dR' = -\frac{1}{2} m\omega^2 R^2 \,.$$

To find the constant (time-indep.) radius R, minimize $U(R) = U_C(R) + U_E(R)$, by setting $\frac{d}{dR} U(R) = 0$:

→ $$R = \frac{R_0}{1 - \frac{m\omega^2}{4\pi^2 k}}.$$

SOLUTION III (Lagrangian Mechanics):

Assuming axial symmetry, the Lagrangian of the spinning ring subjected to elastic force only is

$$L = \frac{m(\dot{R}^2 + R^2\dot{\varphi}^2)}{2} - \frac{k[2\pi(R - R_0)]^2}{2}$$

The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{R}}\right) - \frac{\partial L}{\partial R} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$$

or,

$$m\ddot{R} - mR\dot{\varphi}^2 + 4\pi^2 k(R - R_0) = 0, \qquad 2R\dot{R}\dot{\varphi} + R^2\ddot{\varphi} = 0$$

We are interested in the stationary solution with $\dot{R} = 0$, $\ddot{R} = 0$, and $\dot{\varphi} = \omega$, $\ddot{\varphi} = 0$.

This gives

$$-mR\omega^2 + 4\pi^2 k(R - R_0) = 0$$

and, thus,

$$R = \frac{R_0}{1 - \frac{m\omega^2}{4\pi^2 k}}$$

as before.

SOLUTION IV (Hamiltonian Mechanics):

Assuming axial symmetry, the Lagrangian of the spinning ring subjected to elastic force only is

$$L = \frac{m(\dot{R}^2 + R^2\dot{\varphi}^2)}{2} - \frac{k[2\pi(R-R_0)]^2}{2}$$

The generalized momenta are

$$p_R = \frac{\partial L}{\partial \dot{R}} = m\dot{R}, \qquad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mR^2\dot{\varphi}$$

resulting in the Hamiltonian,

$$H = p_R\dot{R} + p_\varphi\dot{\varphi} - L = \frac{{p_R}^2}{2m} + \frac{{p_\varphi}^2}{2mR^2} + \frac{k[2\pi(R-R_0)]^2}{2}$$

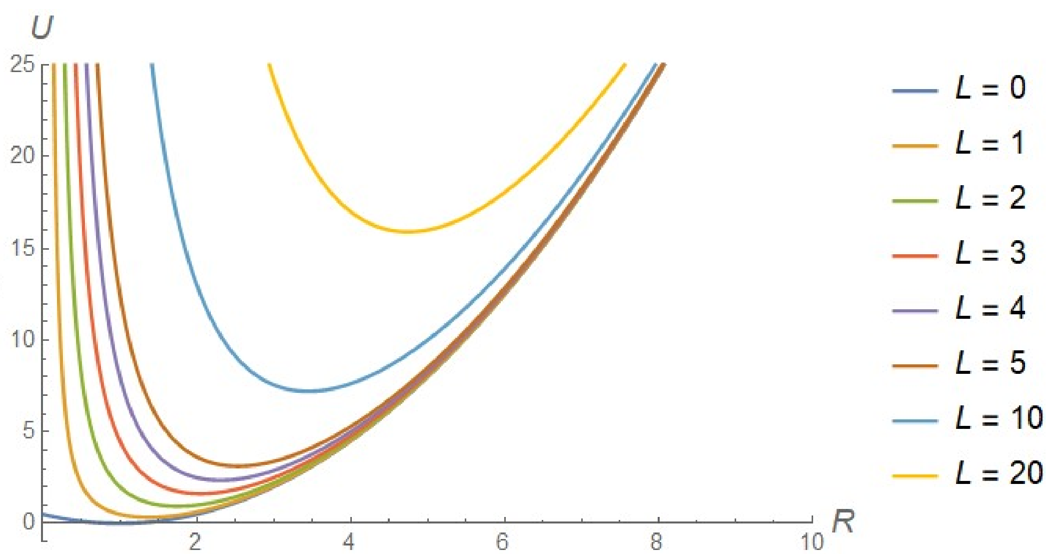
The Hamilton equations of motion immediately show that p_φ (the angular momentum) is conserved (since there is no explicit dependence on φ). Denoting the constant $p_\varphi \equiv l$, we get,

$$H = \frac{{p_R}^2}{2m} + \frac{l^2}{2mR^2} + \frac{k[2\pi(R-R_0)]^2}{2}$$

which is a one-dimensional problem with the effective potential energy,

$$U(R) = \frac{l^2}{2mR^2} + \frac{k[2\pi(R-R_0)]^2}{2}$$

whose graph is given below (with $m = k = R_0 = 1$):



The equilibrium value of R is found by minimizing the $U(R)$, subject to the constraint $l = mR^2\omega = \text{constant}$,

$$\frac{dU(R)}{dR} = -\frac{l^2}{mR^3} + 4\pi^2 k(R - R_0) = 0$$

or,

$$-\frac{(mR^2\omega)^2}{mR^3} + 4\pi^2 k(R - R_0) = 0$$

giving

$$-mR\omega^2 + 4\pi^2 k(R - R_0) = 0$$

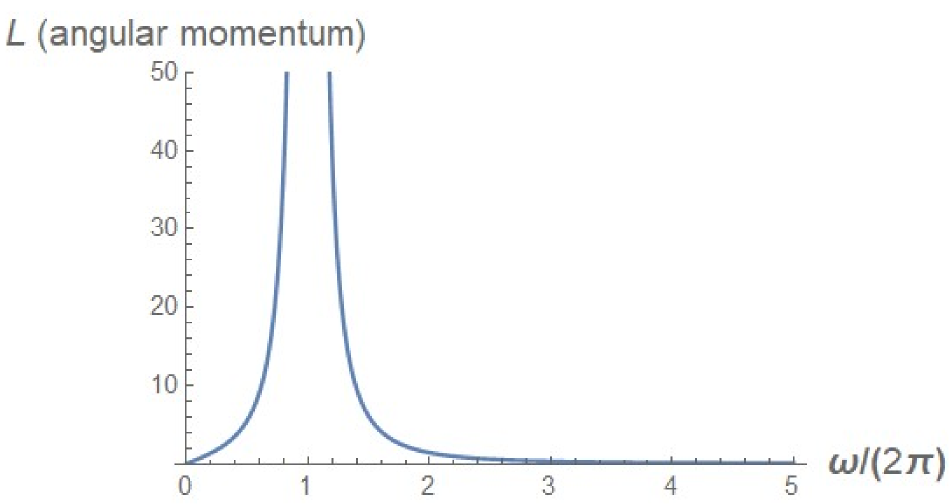
and, thus,

$$R = \frac{R_0}{1 - \frac{m\omega^2}{4\pi^2 k}}$$

as before. Notice, incidentally, that the "equilibrium" angular momentum as a function of ω is given by

$$l(\omega) = mR^2\omega = \frac{m{R_0}^2\omega}{\left(1 - \frac{m\omega^2}{4\pi^2 k}\right)^2}$$

whose graph is given below (with $m = k = R_0 = 1$)



[<u>FULL CREDIT FOR UP TO HERE.</u>]

Units check: $\left[\frac{m\omega^2}{4\pi^2 k}\right] = \frac{\mathrm{kg/s^2}}{\mathrm{N/m}} = 1$ (as required).

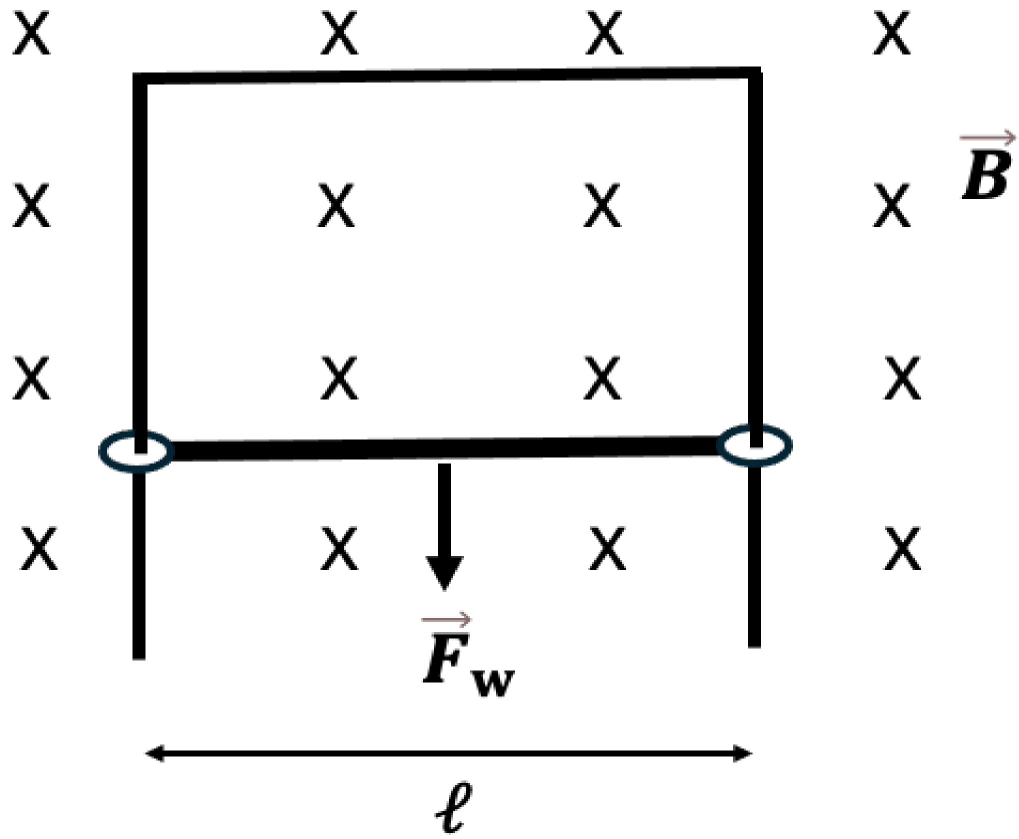
The formula is valid for $\omega < 2\pi\sqrt{k/m}$. At $\omega > 2\pi\sqrt{k/m}$ the ring stretches indefinitely. Physically, however the constraint is even stronger, $\omega \ll 2\pi\sqrt{k/m}$, due to the breakdown of the "Hooke's Law" at large elongations, so

$$R \approx R_0\left(1 + \frac{m\omega^2}{4\pi^2 k}\right).$$

In terms of the Route II solution above, the total potential energy, $U(R)$, has a local minimum and the solution for R above is a stable equilibrium if $\omega < 2\pi\sqrt{k/m}$. If $\omega > 2\pi\sqrt{k/m}$ $U(R)$ is monotonically decreasing and does not have a local extremum for any positive R, *i.e.*, the system is unstable.

Problem 3: Electricity and Magnetism (EM1)

The figure below shows a U-shaped conducting rail that is oriented vertically in a horizontal, uniform magnetic field B. The rail has no electric resistance and does not move. A slide wire with mass m, length ℓ, and resistance R can slide up and down without friction and without air resistance, while maintaining electrical contact with the rail. The slide wire is released from rest, subject to its downward weight force, $\vec{F}_{\mathrm{w}}$, and to the magnetic force. Show that the slide wire reaches a constant terminal speed, v_{term}. State v_{term}, expressed in terms of B, m, ℓ, R and gravitational acceleration g.



SOLUTION to EM1

As the wire falls, the flux into the page will increase. This will induce a current to oppose the increase, so the induced current will flow counterclockwise. As this current passes through the slide wire, it experiences an upward magnetic force. So there is an upward force—a retarding force—on the wire as it falls in the field. As the wire speeds up, the retarding force will become larger until it balances the weight.

The force on the current-carrying slide wire is

$$F_{\mathrm{m}} = I\,\ell\,B.$$

The induced current is

$$I = \frac{\mathcal{E}}{R} = \frac{1}{R}\left|\frac{d\Phi}{dt}\right| = \frac{1}{R}\left|\frac{d}{dt}(AB)\right| = \frac{B}{R}\left|\frac{d}{dt}(\ell x)\right| = \frac{B\ell}{R}\left|\frac{d}{dt}x\right| = \frac{B\ell v}{R}\ .$$

Consequently,

$$F_{\mathrm{m}} = I\,\ell\,B = \frac{\ell^2 B^2}{R} v\ .$$

The magnetic force vector, $\vec{F}_{\mathrm{m}} = I\ \vec{\ell} \times \vec{B}$, points upward, since $\vec{B}$ points into the paper and $\vec{\ell}$ points in the direction of the current flow in the slide wire, *i.e.*, horizontally from left to right.

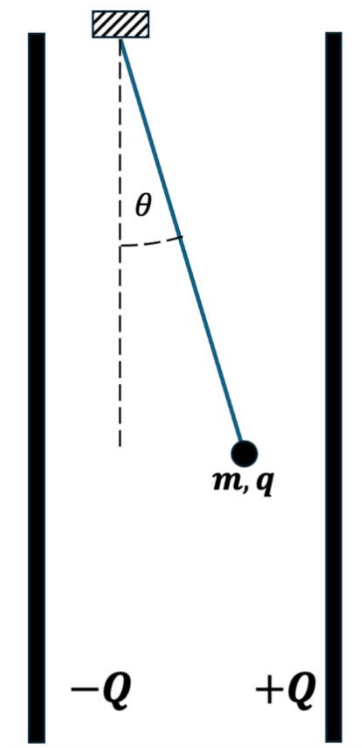
The important point is that F_{m} is proportional to the speed *v*. As the wire begins to fall and its speed increases, so does the retarding magnetic force. Within a very short time, F_{m} will increase in size to where it matches the weight $F_{\mathrm{w}} = mg$. At that point, there is no net force on the slide wire, so it will continue to fall at a *constant speed*. The condition that the magnetic force equals the weight is

$$\frac{\ell^2 B^2}{R} v_{\mathrm{term}} = mg \quad \Rightarrow \quad v_{\mathrm{term}} = \frac{mgR}{\ell^2 B^2}$$

Problem 4: Electricity and Magnetism (EM2)

A charged point particle of mass $m = 6.0\text{g}$ and unknown charge q is suspended in vacuum from a massless string between two parallel planar capacitor plates, carrying charges $\pm Q$, with $Q = +5.0\mu\text{C}$, uniformly spread out over the respective opposing plate surfaces of area $A = 2.0\text{m}^2$ per plate.

The vector of Earth's gravitational force acting on the particle points vertically downward, parallel to the capacitor plate surfaces. The gravitational acceleration is $g = 9.81\ \text{m/s}^2$. In equilibrium, the string is deflected by an angle $\theta = 25^o$ from the vertical towards the right capacitor plate, as shown below.



(a) Find the strength of the electric field $E = |\vec{E}|$ produced between the two capacitor plates by their charges $\pm Q$. State the direction of the electric field vector, $\vec{E}$.

Hint: The permittivity of free space is $\epsilon_o = 8.85 \times 10^{-12}\ \text{C}^2\text{N}^{-1}\ \text{m}^{-2}$.

(b) Find the charge q carried by the point particle.

SOLUTION to EM2

(a) $E = \frac{1}{\epsilon_o}\frac{Q}{A} = \frac{1}{(8.85\times10^{-12}\ \mathrm{C^2N^{-1}\ m^{-2}})}\frac{(5\times10^{-6}\mathrm{C})}{(2\mathrm{m}^2)} = \mathbf{282.49\ kN/C}$

Vector $\vec{E}$ points **horizontally to the left**, from positively charged plate to negatively charged plate.

(b) Force components perpendicular to the string:

Gravitational force $\vec{F}_g$: $F_{g\perp} = mg\ \sin(\theta)$

Electric force $\vec{F}_e$: $F_{e\perp} = |q|\ E\ \cos(\theta)$

Force equilibrium perpendicular to string:

$F_{g\perp} = F_{e\perp}$

→ $|q| = \frac{mg}{E}\ \tan(\theta) = \frac{(0.006\ \mathrm{kg})(9.81\mathrm{ms}^{-2})}{(282.49\times10^3\mathrm{N/C})}\ \tan(25^o) = 97.16\ \mathrm{nC}$

Sign of q:

$\vec{F}_e = q\ \vec{E}$: points horizontally to the right, since string is deflected to the right

$\vec{E}$: points horizontally to the left, opposite to $\vec{F}_e$

→ q must be negative

→ $q = \mathbf{-97.16\ nC}$

The University of Georgia

Department of Physics and Astronomy

Prelim Exam

August 9, 2024

Part II (Problems 5 and 6)

3:00 pm – 5:00 pm

Instructions:

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- Write your work and reasoning in a neat, clear, and logical manner so that the grader can follow it. Lack of clarity is likely to result in a substantial penalty.

Problem 5: Quantum Mechanics (QM1)

Compton realized in 1921 that, if X-rays with incident frequency f hit approximately free electrons in a solid material, the electrons do not only emit X-rays of the same frequency. He also observed another X-ray signal, emitted with frequency $f' \leq f$. The experiments revealed further that the difference of the respective wavelengths,

$$\Delta\lambda = c/f' - c/f \ ,$$

depends only on the scattering angle, θ (defined below), but $\Delta\lambda$ is independent of the incident X-ray frequency, f, and independent of the target material. Here, c denotes the speed of light.

In 1923, Compton himself explained this effect on the basis of Einstein's hypothesis that X-rays of frequency f consist of photons which can be considered as particles with relativistic energy $E_\gamma = hf$, momentum $\vec{p}_\gamma$, and zero rest mass.

(a) Write down the relativistic conservation equations for the total relativistic energy and momentum before and after a photon-electron collision, with electron momentum $\vec{p}_{\mathrm{e}} = 0$ before the collision.

(b) Then determine $\Delta\lambda$ as a function of the angle θ between incident and emitted X-rays.

Hint: The relativistic energy of *any* free particle of momentum $\vec{p}$ and rest mass m is

$$E = \sqrt{m^2c^4 + c^2|\vec{p}|^2} \ .$$

(c) Calculate $\Delta\lambda$ specifically for $\theta = \pi/2$ ("Compton wavelength of the electron").

Constants:

Planck's constant: $h \approx 6.62607 \times 10^{-34}$ Js

Electron mass: $m_e \approx 9.10938 \times 10^{-31}$ kg

Speed of light: $c \approx 2.99792 \times 10^{8}$ m/s

SOLUTION to QM1

Quantum Mechanics

Problem:
Compton realized in 1921 that if X-rays with frequency ν hit quasi-free crystal electrons, excited electrons do not only emit light of the same frequency. He also observed another signal with frequency $\nu' \leq \nu$. The experiments revealed further that the difference of the respective wavelengths $\Delta\lambda = c/\nu' - c/\nu$ (c: speed of light) depends only on the scattering angle, but is independent of the target material.

In 1923, Compton himself explained this effect on the basis of Einstein's hypothesis that photons can be considered as particles with energy $h\nu$, momentum $\boldsymbol{p}_\gamma$, and zero rest mass.

Write down the balance equations (relativistic problem!) for the total energy and momentum before ($\boldsymbol{p}_e = \boldsymbol{0}$) and after the collision. Determine $\Delta\lambda$ as a function of the angle θ between incident and emitted light. Calculate $\Delta\lambda$ specifically for $\theta = \pi/2$ ("Compton wavelength of electron").

Constants:
$h \approx 6.62607 \times 10^{-34}$ Js
$m_e \approx 9.10938 \times 10^{-31}$ kg
$c \approx 2.99792 \times 10^{8}$ m/s

Solution:
Photon energy: $E_\gamma = h\nu = |\boldsymbol{p}_\gamma|c$, photon momentum: $\boldsymbol{p}_\gamma = \frac{h\nu}{c}\boldsymbol{n}$ with $\boldsymbol{n}$: unit vector of direction of photon propagation, electron energy: $E_e = \sqrt{m_e^2c^4 + \boldsymbol{p}_e^2c^2}$

Energy balance:
before collision: $E = E_\gamma + E_e = h\nu + m_ec^2$
after collision: $E' = E'_\gamma + E'_e = h\nu' + \sqrt{m_e^2c^4 + \boldsymbol{p}_e'^2c^2}$
balance: $E = E'$

Momentum balance:
before: $\boldsymbol{p} = \boldsymbol{p}_\gamma + \boldsymbol{p}_e = \frac{h\nu}{c}\boldsymbol{n}$
after: $\boldsymbol{p}' = \boldsymbol{p}'_\gamma + \boldsymbol{p}'_e = \frac{h\nu'}{c}\boldsymbol{n}' + \boldsymbol{p}'_e$
balance: $\boldsymbol{p} = \boldsymbol{p}'$

Thus, $\boldsymbol{p}'_e = \frac{h}{c}(\nu\boldsymbol{n} - \nu'\boldsymbol{n}')$. Substituting square $\boldsymbol{p}_e'^2 = \frac{h^2}{c^2}(\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta)$, where θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$, in energy balance yields $1/\nu' - 1/\nu = \frac{h}{m_ec^2}(1 - \cos\theta)$. Thus, $\Delta\lambda = \frac{h}{m_ec}(1 - \cos\theta)$.

For $\theta = \pi/2$, it is $\Delta\lambda = \frac{h}{m_ec} \approx 2.426 \times 10^{-12}$m.

Problem 6: Quantum Mechanics (QM2)

(a) Write down the Hamiltonian for a quantum particle (moving on the x axis) of mass *m* and charge *q* in a one-dimensional harmonic oscillator potential and a uniform electric field *E* in the $+x$-direction. The harmonic oscillator frequency is ω.

State your result in terms of m, ω, q, E and in terms of the operators $\hat{x}$ and $\hat{p}$, representing the particle's position and momentum.

(b) Evaluate the Heisenberg equations of motion for $\hat{x}$ and $\hat{p}$.

Hint: In the Heisenberg picture of quantum mechanics, the operators representing observables are time-dependent and the wave functions are time-independent. Given a time-independent Hamiltonian, $\widehat{H}$, the time-dependent operator of any observable, $\hat{Q}(t)$, obeys the Heisenberg equation of motion

$$\frac{d}{dt}\hat{Q}(t) = +\frac{i}{\hbar}\left[\widehat{H}\,,\ \hat{Q}(t)\,\right]$$

where $\left[\hat{A}\,,\hat{B}\,\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ denotes the commutator, for any two operators $\hat{A}$ and $\hat{B}$ and $\hbar = h/(2\pi)$ and h is Planck's constant. Evaluate the right-hand side of this equation for $\hat{x}$ and then for $\hat{p}$, each expressed in terms of m, ω, q, E, $\hat{x}$ and/or $\hat{p}$. Assume the commutator of $\hat{x}$ and $\hat{p}$ is time-independent.

(c) Show that the Heisenberg equations of motion of $\hat{x}$ and $\hat{p}$ are operator versions of the classical equations of motion for the classical dynamical variables of position and momentum, *x* and *p.*

SOLUTION to QM2

(a) Kinetic energy operator:

$$\hat{T} = \frac{1}{2m}\hat{p}^{\,2} .$$

Potential energy operator:

$$\hat{V} = \frac{m\,\omega^2}{2}\hat{x}^2 - qE\hat{x} \ .$$

The Hamiltonian is $\hat{H} = \hat{T} + \hat{V}$:

→ $$\hat{H} = \frac{1}{2m}\hat{p}^{\,2} + \frac{m\,\omega^2}{2}\hat{x}^2 - qE\hat{x} .$$

(b) Equations of motion are

[1] $$\frac{d}{dt}\hat{x} = \frac{i}{\hbar}\left[\hat{H},\hat{x}\right] = \frac{1}{m}\hat{p}$$

[2] $$\frac{d}{dt}\hat{p} = \frac{i}{\hbar}\left[\hat{H},\hat{p}\right] = -m\omega^2\hat{x} + qE .$$

(c) Classically:

$$\frac{d}{dt}x = v = \frac{1}{m}p$$ → Compare to [1]

$$V = \frac{m\,\omega^2}{2}x^2 - qEx$$

$$F = -\frac{dV}{dx} = -m\omega^2 x + qE$$

$\frac{d}{dt}p = F = -m\omega^2 x + qE$ → Compare to [2]